A Hybrid KP-ABE Scheme for Universal Circuits, Based on Monotone Span Programs

Master's Thesis in Computer Science

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Introduction

Overview of Our Work

Introduction

- **Semester 1:** Studying the theoretical foundations of (pairing-based) attribute-based encryption (ABE), monotone access structures (MASes), and monotone span program (MSP) constructions.
- **Semester 2:** Studying KP-ABE schemes for monotone Boolean circuits (MBCs) and MSP lower bounds; optimizing the MSP construction from MBCs with backtracking. $\Rightarrow$ *Paper rejected at Secrypt.*
- **Semester 3:** Studying graph access structures (GASes) and ABE libraries; introducing `U`-gates; proving a new lower bound. $\Rightarrow$ *Paper accepted at CSJM!*
- **Semester 4:** Designing a hybrid KP-ABE scheme for universal circuits. $\Rightarrow$ *Paper rejected at SAC, submitted at WISA.*

Motivation (1 / 3)

Introduction

- Cloud computing has become the backbone of modern enterprise software.
- **Privacy Concerns:** In typical cloud setups, service providers may have unrestricted access to sensitive data.
- **Solution:** Key-policy ABE (KP-ABE) and ciphertext-policy ABE (CP-ABE).
- In KP-ABE, files are encrypted based on their attributes, such that only those users whose access policies are satisfied by these attributes can decrypt them.

Motivation (2 / 3)

Introduction

- There is a need for KP-ABE schemes running on more and more expressive access policies.
- More compact policies $\Rightarrow$ Shorter decryption keys.
- Trees $\rightarrow$ Circuits.
- `AND`/`OR`-gates [1] $\rightarrow$ (t/n)-threshold gates [2] $\rightarrow$ "CAS-nodes" [3] $\rightarrow$?

Motivation (3 / 3)

Introduction

- All state-of-the-art schemes [1, 4] for monotone Boolean circuits (MBCs) yield exponentially large keys, in order not to sacrifice the scheme security.
- Most existing results regarding ABE, linear secret sharing schemes (LSSSes), and monotone span programs (MSPs) are not clearly correlated in the literature.

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Preliminaries

Secret Sharing

Preliminaries

- A secret is distributed among a group of participants.
- Only specific subsets of participants can reconstruct the secret.

Monotone Access Structures

Preliminaries

- Let $\mathcal{P}$ be the set of participants.
- $\mathbb{A} \subseteq 2^{\mathcal{P}}$.
- **Monotonicity:** $\mathcal{X} \in \mathbb{A} \wedge \mathcal{X} \subseteq \mathcal{Y} \rightarrow \mathcal{Y} \in \mathbb{A}$.

Monotone Access Structures — Example

- $\mathbb{A} \equiv ((A \vee B) \wedge (B \vee C))$.
- $\mathbb{A} = \{\{B\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}$.

Monotone Span Programs

Preliminaries

- $\hat{M} = (M, \rho)$:
  - M is an $m \times d$ matrix.
  - $\rho : \{1, \ldots, m\} \to \mathcal{P}$ is a labeling function.
- $\hat{M}$ **computes** $\mathbb{A}$ if $\vec{1} \in \text{span}(M_{\mathcal{X}}) \leftrightarrow \mathcal{X} \in \mathbb{A}$.
- The **size** of $\hat{M}$ is m.
- **Theorem:** $\hat{M}$ never needs more than m columns to compute $\mathbb{A}$ [5].

Monotone Span Programs — Example

Preliminaries

- $\mathbb{A} \equiv ((A \wedge B) \vee (B \wedge C))$.
- $\vec{1} = (1, 0)$.
- $\hat{M} = \left[\begin{array}{c|cc} A & 0 & -1 \\ B & 1 & 1 \\ C & 0 & -1 \end{array}\right]$.

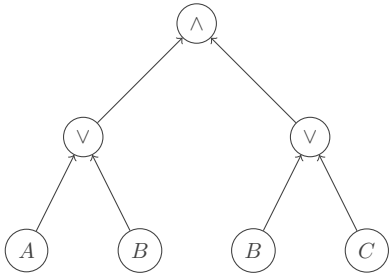
Linear Secret Sharing Schemes

Preliminaries

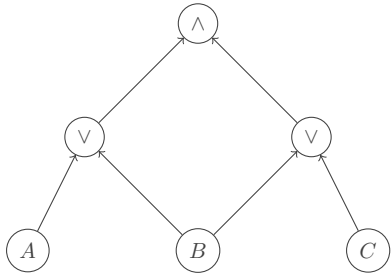
- Each share is given in the form of one or multiple vectors.
- The reconstruction of the secret involves only linear operations over them.
- The **size** of an LSSS is the number of shared vectors.
- An LSSS is **ideal** if each share consists of exactly one vector.
- **Theorem:** LSSSes and MSPs are equivalent [6].

Computational Models

Preliminaries



(a) An MBT computing $\mathbb{A}$.



(b) An MBC computing $\mathbb{A}$.

Figure: An MBT and, respectively, an equivalent MBC, both of them computing the same MAS $\mathbb{A} = \{\{B\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}$.

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Monotone Span Program Limitations

Operations, Constructions, Lower Bounds

Monotone Span Program Limitations

- There are known operations that combine MSPs [7].
- There are known constructions for MSPs from MBTs of in-degree 2 [8] and from threshold trees [9].
- There are known MSP lower bounds [10, 11, 12].

Theoretical Lower Bounds

Monotone Span Program Limitations

- **Theorem:** A lower bound for the size of an MSP computing a particular **monotone access structure** (in *the number of participants*) is $2^{\Omega(n)}$ [13].
- **Problem:** Lack of practical character, since the size of the input to the LSSS is not actually n.

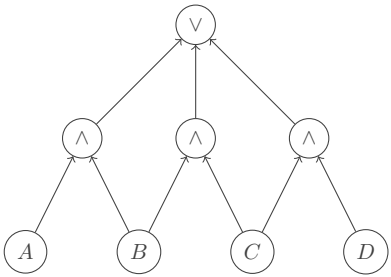
Practical Lower Bounds

Monotone Span Program Limitations

- **Question:** What about a lower bound for the size of an MSP computing a particular **monotone Boolean circuit** (in *its size*)?
- **Proved Claim:** Some polynomial-size monotone Boolean circuits require exponential-size MSPs.

U-Gates

Monotone Span Program Limitations



- $\mathbb{A} \equiv ((A \wedge B) \vee (B \wedge C) \vee (C \wedge D))$.
- Does not admit an MSP of size 4 (proved).
- Can be computed by an MSP of size 5.
- $\vec{1} = (1, 0, 0)$.
- $\hat{M} = \left[\begin{array}{c|ccc} A & 0 & -1 & 0 \\ B & 1 & 1 & 0 \\ C & 0 & -1 & 0 \\ C & 1 & 0 & 1 \\ D & 0 & 0 & -1 \end{array}\right]$.

Graph Access Structures

Monotone Span Program Limitations

- Let $\mathcal{P}$ be the set of participants.
- Let $\mathbb{A}^+$ be the set of minterms in $\mathbb{A}$.
- $\mathbb{A}^+ \subseteq \{\{U, V\} \mid U, V \in \mathcal{P}, U \neq V\}$.

Criterion for Graphs to Admit Ideal LSSSes

Monotone Span Program Limitations

- They should not contain `U`-gates as subgraphs.
- Equivalently, they should be multipartite.
- It is also a sufficient condition [14].

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From Monotone Boolean Circuits to KP-ABE Schemes

Construction Approaches

From Monotone Boolean Circuits to KP-ABE Schemes

1. Circuit $\xrightarrow{[4]}$ LSSS.
2. Circuit $\xrightarrow{\text{NIM optimization [15]}}$ Circuit $\xrightarrow{[4]}$ LSSS.
3. Circuit $\rightarrow$ Tree $\rightarrow$ 2-Tree $\xrightarrow{[8]}$ MSP $\xrightarrow{[2]}$ LSSS.
4. Circuit $\rightarrow$ Tree $\rightarrow$ DNF-Tree $\equiv$ Access Structure $\xrightarrow{\text{backtracking}}$ MSP $\xrightarrow{[2]}$ LSSS.

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A Hybrid KP-ABE Scheme for Universal Circuits

Universal Circuits

A Hybrid KP-ABE Scheme for Universal Circuits

- A directed acyclic graph consisting of:
  1. *input* (i.e., of in-degree zero) nodes, which embed Boolean variables over the attribute set;
  2. *internal* (i.e., of in-degree non-zero) nodes, which embed MSPs computing specific local MASes;
  3. exactly one *output* (i.e., of out-degree zero) node, indicating whether the overall access policy is fulfilled.

Example — Universal Circuit

A Hybrid KP-ABE Scheme for Universal Circuits

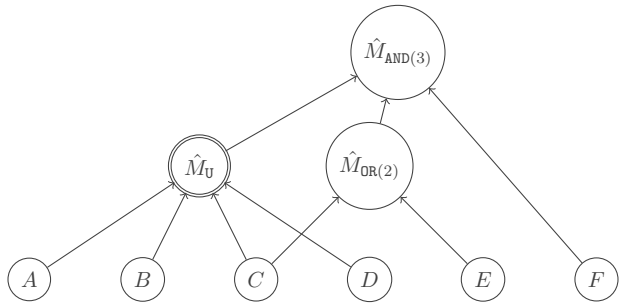

Figure: Example of a universal circuit $\tilde{\mathcal{C}}$ using `AND`, `OR`, and `U`-gates.

Example — Monotone Boolean Circuit

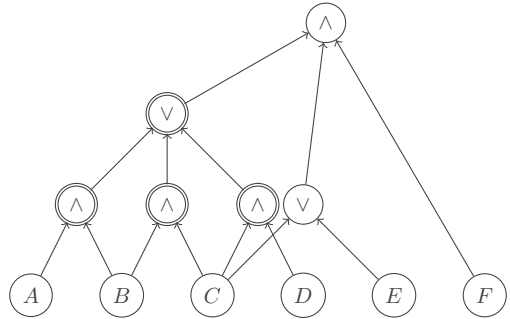

Figure: The circuit $\tilde{\mathcal{C}}$ rewritten as a monotone Boolean circuit $\tilde{\mathcal{C}}'$.

Implementation Detail

A Hybrid KP-ABE Scheme for Universal Circuits

- The implementation does not rely on the entire matrix of each MSP $\hat{M}$, but rather on two custom associated functions:
  1. one that computes the dot product between M and a given vector $\vec{r}$;
  2. one that finds solutions to the equation $\vec{\alpha} \cdot M_{\mathcal{X}} = \vec{1}$.
- We provide the optimized implementations for the `AND`, `OR`, and (t/n)-threshold gates.

Construction

A Hybrid KP-ABE Scheme for Universal Circuits

- Highly technical.
- Based on the constructions of the Hu–Gao scheme for Boolean circuits [1] and the Goyal et al. scheme for MSPs [2].
- Like Hu–Gao, we structure the decryption process over a circuit.
- Over the MSP of each gate, we apply the Goyal et al. scheme.

Security Proof

A Hybrid KP-ABE Scheme for Universal Circuits

- Highly technical.
- Reduction from the Decisional Bilinear Diffie–Hellman problem to the security game.
- Based on the proofs of the Goyal et al. scheme for threshold trees and of the Goyal et al. scheme for MSPs [2].

Efficiency

A Hybrid KP-ABE Scheme for Universal Circuits

- For Boolean circuits and threshold trees — as efficient as previous existing schemes.
- For strictly universal circuits — more efficient, since they can encode policies using way fewer gates, resulting in way lower decryption key sizes.
- Downside inherited from Hu–Gao — the decryption key size may be exponential in the size of the circuit, since the key size equals the number of leaves of the DFS-tree.

Example — Monotone Boolean Circuit

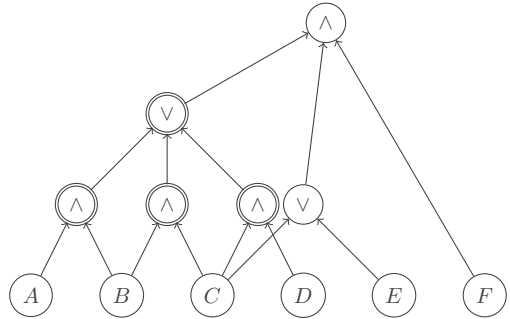

Figure: The circuit $\tilde{\mathcal{C}}$ rewritten as a monotone Boolean circuit $\tilde{\mathcal{C}}'$.

Example — Monotone Boolean Tree

A Hybrid KP-ABE Scheme for Universal Circuits

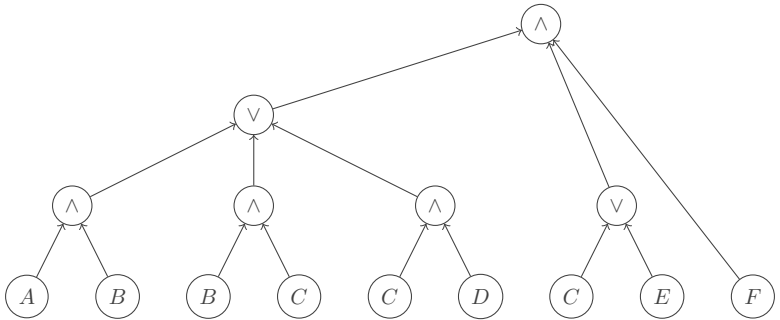

Figure: The monotone Boolean circuit $\tilde{\mathcal{C}}'$ rewritten as a monotone Boolean tree (its DFS-tree) $\tilde{\mathcal{T}}'$.

Implementation

A Hybrid KP-ABE Scheme for Universal Circuits

- Openly accessible on GitHub.
- Written in `C++` (for speed) and `Python` (for linear algebra).
- Using the `C++` wrapper of the `PBC` library for pairing operations.
- Using the `Galois` and `NumPy` libraries for computing $M_{\mathcal{X}'}^{-1}$.
- Developers can define fully-customizable MSP classes, just by writing custom `dot` and `solve` functions.
- The MSP implementations for the `AND`, `OR`, and (t/n)-threshold gates are already provided and highly optimized.

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Conclusion

Our Contribution

Conclusion

- Provided a novel way of proving the nonexistence of ideal LSSSes for certain MASes.
- Proved a new MSP lower bound.
- Introduced the concept of “universal circuits” — circuits at the highest degree of generalization.
- Developed a pairing-based KP-ABE scheme for them.
- Proved that it is more efficient than the Hu–Gao scheme for “general circuits” and the Goyal et al. scheme for “access trees.”
- Implemented it in `C++` (on GitHub).

Future Work

Conclusion

- The decryption key size may be exponential in the size of the circuit — room for improvement.
- More MSP-embedded gates are to be created and implemented (e.g., for "CAS-nodes" [3]).
- Improving the implementation by transpiling the `Python` code for inverting a `Galois` matrix into `C++`.

Bibliography

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Q & A

Thank you for your attention!